Finite temperature Functional RG, droplets and decaying Burgers Turbulence

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Abstract. – The functional RG (FRG) approach to pinning of d-dimensional manifolds is reexamined at any temperature T. A simple relation between the coupling function $R(u)$ and a physical observable is shown in any d. In $d = 0$ its beta function is displayed to a high order, ambiguities resolved; for random field disorder (Sinai model) we obtain exactly the $T = 0$ fixed point $R(u)$ as well as its thermal boundary layer (TBL) form (i.e. for $u \sim T$) at $T > 0$. Connection between FRG in $d = 0$ and decaying Burgers is discussed. An exact solution to the functional RG hierarchy in the TBL is obtained for any d and related to droplet probabilities.

Elastic manifolds pinned by quenched disorder [1] are the simplest system to study glass phases where (dimensionless) temperature is formally irrelevant, scaling as $\tilde{T}_L = TL^{-\theta}$ with system size L. They are parameterized by a displacement (N-component height) field $u(x) \equiv u_x$, where x spans a d-dimensional internal space. The competition between elasticity and disorder produces rough ground states with sample averages $\overline{(u_x - u_{x'})^2} \sim |x - x'|^{2\zeta}$ ($\theta = d - 2 + 2\zeta$). These are believed to be statistically scale invariant, hence should be described by a critical (continuum) field theory (FT). The latter seems highly unconventional in several respects. First, an infinite number of operators become marginal simultaneously in $d = 4 - \epsilon$. This is handled via Functional RG methods where the relevant coupling constant becomes a function of the field, $R(u)$, interpreted as the (second cumulant) disorder correlator [2]. A more formidable difficulty then arises: at $T = 0$ both $R(u)$ and, more generally, the full effective action functional $\Gamma[u]$, appear to be non-analytic ([1]) around $u = 0$. A linear cusp in $R''(u)$ was found in one loop and large N calculations [2, 3]. Qualitative (two mode minimization) and mean field arguments relate this cusp to multiple metastable states and shock type singularities in the energy landscape [4]. As a consequence, ambiguities arise in loop corrections [5]. Although candidate renormalizable FTs have been identified [5, 6] (working directly at $T = 0$) this problem has, until now, hampered derivation of the field theory from first principles (with the notable exception of the $N = 1$ depinning transition [7]).

Working at non-zero temperature $T > 0$ should help define the theory, and $\Gamma[u]$ has been argued to remain smooth within a "thermal boundary layer" (TBL) of width $u \sim \tilde{T}_L$ around $u = 0$. This width however shrinks as $\tilde{T}_L \to 0$ in the thermodynamic limit, and if a fixed u, large L limit exists for any fixed small T it should unambiguously define the (non-analytic)

([1])the non-analyticity of Γ occurs at *finite scale* (the Larkin length) contrarily to e.g. the critical ϕ^4 theory

"zero temperature theory": this program, called "matching", was proposed and extensively studied in Ref. [8]. It does, to some extent, rely on a scaling ansatz proposed there for the TBL. This ansatz was shown to be consistent to 1-loop with the droplet picture [8] and, in the (near) equilibrium (driven) dynamics, to account for the phenomenology of ultraslow activated (creep) motion [8,9]. Although its physics is reasonable, it is not yet established how a critical renormalizable FT emerges from it as $\tilde{T}_L \to 0$, with a finite unambiguous beta function.

Another field of physics where an (unconventional) field theoretic description is needed, but remains elusive, is high Reynolds number turbulence. There too the scale invariant regime, the inertial range, needs regularization at the small dissipation scale set by the (formally irrelevant) viscosity ν [10]. Connections between these two tantalizing problems can be made quantitative within the simplified Burgers turbulence, a much studied problem [11–14].

Given its central role in the FRG, it is of high interest to obtain the *precise* physical content of the (fixed point) function $R(u)$, beyond previous qualitative arguments. In the FT, a precise, but abstract, definition was given, from the replicated effective action at zero momentum, which allowed for a systematic dimensional expansion. From it, it was observed that $R''(0)$ gives the exact sample to sample variance of the center of mass of the manifold (a typical observable with a universal $T = 0$ limit), while $R''''(0)$ yields sample to sample susceptibility fluctuations (a finite temperature observable which diverges as $T_L \to 0$). It would be useful to relate directly the full function $R(u)$ to an observable and cleanly separate zero from finite T contributions.

The present Letter is a short account of a recent study [15] aimed at clarifying the physics encoded in the FRG and its connections to Burgers turbulence. We obtain a simple operational definition valid in any d, not only for $R(u)$, but also for higher cumulants, and the full (replicated) effective action $\Gamma[u]$. It makes explicit its $T = 0$ physics and at $T > 0$ makes precise the relation between the TBL form of the effective action, droplet probabilities, and dilute (functional) shocks, via a (functional) decaying Burgers equation. Next, the instructive $d = 0$ case is studied. For $N = 1$, the matching program started in Ref. [8] is pushed to obtain here the (unambiguous) beta function to four loop, and related to works on the inviscid distributional limit of Burgers equation [11]. In the sub-case of the Sinai (i.e. random field) model, the exact $R(u)$ is computed at $T = 0$. The TBL rounding form at $T > 0$ is also obtained. Obtaining the thermal rounding form in any d amounts to solve an infinite hierarchy of (functional) exact RG equations: remarkably, this can be achieved, the solution being parameterized by droplet probability data. All details are given in [15].

The model studied here is defined by the total energy:

$$H_V[u] = \frac{1}{2}\int_{xy} g^{-1}_{xy} u_x u_y + \int_x V(u_x, x) \tag{1}$$

in a given sample ($u \in R^N$). The distribution of the random potential is translationally invariant, with second cumulant $\overline{V(u,x)V(u',x')} = \delta^d(x-x')R_0(u-u')$ and $\overline{V(u,x)} = 0$. This implies the statistical (tilt) symmetry (STS) under $(x, u_x) \to (x, u_x + \phi_x)$ [1]. Several results here are valid for arbitrary g_{xy}, but we often specialize to $g^{-1}_q = q^2 + m^2$ in Fourier space, where the small mass provides a confining parabolic potential and a convenient infrared cutoff at large scale $L_m = 1/m$. In all formula below one can replace $\int_x \equiv \int d^d x \to \sum_x$ and $\delta(x - x') \to \delta_{xx'}$ in the bare disorder correlator, i.e. a lattice provides a UV cutoff which preserves STS Numerous physical systems are modelled by (1) e.g.: (i) CDW or vortex lattices [1], for a periodic $R_0(u)$, (ii) magnetic interfaces with bond disorder, for a short range (SR) $R_0(u)$, (iii) magnetic interfaces with random field disorder, of variance σ, for a long range $R_0(u) \sim -\sigma|u|$ at large u.

Let us briefly recall the convenient definition of $R(u)$ used in the FT. The model is studied using replica fields u_x^a, $a = 1, .., p$, with bare action:

$$\mathcal{S}[u] = \frac{1}{2T} \sum_a \int_{xy} g_{xy}^{-1} u_x^a u_y^a - \frac{1}{2T^2} \sum_{ab} \int_x R_0(u_x^a - u_x^b) \tag{2}$$

and disorder-averaged correlations of (1) identify with replica correlation functions of (2) at $p = 0$. These (the connected ones) are obtained from Taylor expanding the W functional, $W[j] = \ln \int \prod_{ax} du_x^a e^{\int_x \sum_a j_x^a u_x^a - \mathcal{S}[u]}$. From it one defines, via a Legendre Transform, the effective action of the replica theory, $\Gamma[u] = \int_x \sum_a u_x^a j_x^a - W[j]$. It generates (in a Taylor expansion in u) the renormalized vertices, i.e. those where loops have been integrated, and is thus the important functional for the FRG. To define the renormalized disorder one assumes an expansion in number of replica sums:

$$\Gamma[u] = \sum_a \int_{xy} \frac{g_{xy}^{-1} u_x^a u_y^a}{2T} - \sum_{ab} \frac{R[u^{ab}]}{2T^2} - \sum_{abc} \frac{S[u^{abc}]}{3! T^3} + \cdots \tag{3}$$

where STS implies that the single-replica term is the bare one, and the form of the n-replica terms, e.g. $R[u^{ab}]$ is a functional depending only on the field $u_x^{ab} \equiv u_x^a - u_x^b$, whose value for a uniform field (i.e. local part) defines $R(u)$, i.e $R[\{u_x^{ab} = u\}] = L^d R(u)$ ([2]). It was used in the FT [5,6] to compute the beta function, $-m\partial_m|_{R_0} R(u) = \beta[R](u)$, in powers of R, and its derivatives.

We have shown that this abstract definition is equivalent to a physical one: for each realization of the random potential V, one defines the renormalized potential functional $\hat{V}[v] = \hat{V}[\{v_x\}]$ as the free energy of the system when centering the quadratic potential around $u_x = v_x$:

$$\hat{V}[\{v_x\}] = -T \ln \int \prod_x du_x \exp(-H_{V,v}[u]/T) \tag{4}$$

$$H_{V,v}[u] = \frac{1}{2} \int_{xy} g_{xy}^{-1} (u_x - v_x)(u_y - v_y) + \int_x V(u_x, x)$$

Using STS one sees that the renormalized energy landscape has second cumulant correlations:

$$\overline{\hat{V}[\{v_x\}] \hat{V}[\{v'_x\}]} = \hat{R}[\{v_x - v'_x\}] \tag{5}$$

and $\overline{\hat{V}} = 0$ (averages are w.r.t. V). The result shown in [15] is that $\hat{R} = R$. Hence one can *measure* the 2-replica part of the effective action by computing the free energy in a well whose position is varied. Choosing a uniform $v_x = v$, one obtains its local part:

$$\overline{\hat{V}(v) \hat{V}(v')} = L^d R(v - v') \tag{6}$$

where $\hat{V}(v) = \hat{V}[\{v_x = v\}]$, using a parabolic potential centered at $u_x = v$. Performing the Legendre transform [15] (more involved) relations are found for higher cumulants, e.g. $S = \hat{S} - 3 sym_{abc} g_{xy} R'_x[v^{ab}] R'_y[v^{ac}]$. The STS property was used: for a non STS model, e.g. with discrete u, either it flows to the STS fixed point as $m \to 0$, and the above holds asymptotically, or it does not and a (more involved) extension holds [15].

([2])such zero momentum renormalization conditions are standard in a massive theory. It is not presently known how to close FRG using other conditions, e.g. symmetric external momenta, as in massless theories

From (4) the *renormalized (pinning) force* functional $\mathcal{F}_x = \hat{V}'_x[v] \equiv \delta\hat{V}[v]/\delta v_x$ is related to the thermally averaged position in presence of the shifted well, via $\mathcal{F}_x = \int_y g^{-1}_{xy}(v_y - \langle u_y\rangle_{H_{V,v}})$. Hence the force correlator functional $R''_{xy}[v]$ has a nice expression. For uniform $v_x = v$ and at $T = 0$ it is simple: denote $u_x(v)$ the minimum energy configuration of $H_{V,v}[u]$ for a fixed $v_x = v$ and $\bar{u}(v) = L^{-d}\int_x u_x(v)$ its center of mass position. Then, denoting $\Delta(v) = -R''(v)$:

$$\overline{(v - \bar{u}(v))(v' - \bar{u}(v'))} = \Delta(v - v')L^{-d}m^{-4} \tag{7}$$

which generalizes to non-zero T (replacing $\bar{u}(v)$ by its thermal average) and to the full multi-local functional ([3])

For fixed L/a, where a is the UV cutoff scale, the minimum is expected unique for continuous distributions of V, except for a discrete set of values v_s which are positions of shocks where $\bar{u}(v)$ switches between different values (e.g. u_1 to u_2) and the force is discontinuous (at $T = 0$): below, the strength of each shock is noted $u^{(s)}_{21} = u_2 - u_1$.

The renormalized pinning force satisfies an exact RG (ERG) equation (with $\partial g = -m\partial_m g$):

$$-2m\partial_m \mathcal{F}_x[v] = \int_{yz} \partial g_{yz}(T\mathcal{F}''_{xyz}[v] - \mathcal{F}'_{xy}[v]\mathcal{F}_z[v]) \tag{8}$$

a functional generalization of the *decaying Burgers equation* to which it reduces for $d = 0$:

$$\partial_t F(v) = \frac{T}{2}F''(v) - F'(v)F(v) \tag{9}$$

with $t = m^{-2}$, $F(v) = \hat{V}'(v)$, usually written $\partial_t \mathsf{u} + \mathsf{u}'_x\mathsf{u} = \nu\mathsf{u}''_{xx}$, identifying u, x, ν in Burgers to $F, v, T/2$ in the FRG (while $\hat{V}[v]$ satisfies a functional KPZ-type equation). The stochasticity in (8),(9) comes from their (random) initial conditions $F(v) = V'(v)$ and $\mathcal{F}_x[v] = V'_x[v]$ for $t = 0, m = \infty$. Eq. (9) (and its primitive) is equivalent to an infinite ERG hierarchy for the n-th moments $\bar{S}^{(n)}(v_{1,2,..,n}) = (-)^n\overline{\hat{V}(v_1)..\hat{V}(v_n)}$ in $d = 0$:

$$-m\partial_m R(v) = \frac{2T}{m^2}R''(v) + \frac{2}{m^2}\bar{S}_{110}(0,0,v) \tag{10}$$

$$-m\partial_m \bar{S}^{(n)}(v_{1,2,..,n}) = \frac{nT}{m^2}[\bar{S}^{(n)}_{20..0}(v_{1,2,..n})] + \frac{n}{m^2}[\bar{S}^{(n+1)}_{110..0}(v_{1,1,2..n})] \tag{11}$$

where $\bar{S} \equiv \bar{S}^{(3)} = \hat{S}$, subscripts denote partial derivatives and [..] is symmetrization. A similar, more formidable looking functional hierarchy exists for any d:

$$-m\partial_m R[v] = T\partial g_{xy}R''_{xy}[v] + \partial g_{zz'}\bar{S}^{110}_{zz'}[0,0,v] \tag{12}$$

together with ERG equations for $\bar{S}$ and higher moments. In both cases a related hierarchy exists for the cumulants R, S, .. defining $\Gamma[u]$ in (3), studied in [8,16]. The usual RG strategy is to truncate them to a given order in R yielding the beta function. Ambiguities in the limit of coinciding arguments in (11,12) may arise in doing so directly at $T = 0$.

We start with $d = 0$ (and $N = 1$), a particle in a 1D random potential $V(u)$, aiming to obtain an unambiguous beta function as $m \to 0$. We define rescaled $\tilde{T} = 2Tm^\theta$ and $R(u) = \frac{1}{4}m^{\epsilon-4\zeta}\tilde{R}(um^\zeta)$ [this should yield a FP when correlations of V grow as $u^{\theta/\zeta}$]. Trying first standard loop expansion at $T > 0$ ($\tilde{R}$ analytic), we obtained from (11) the beta function

([3])(6), (7) generalize to any N, and to two copies as used in chaos studies [17] $\overline{\hat{V}_i(v)\hat{V}_j(v')} = L^d R_{ij}(v - v')$.

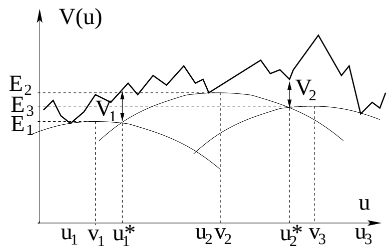

Fig. 1 – Construction of the joint probability $P(\{E_i, v_i\})$ that $\hat{V}(v_i) = E_i$ at points v_i: the random walk $V(u)$ must remain above all parabola centered on the v_i of apex E_i intersecting at points u_i^*. Each independent interval $[u_i^*, u_{i+1}^*]$ can be treated as in [18].

$-m\partial_m \tilde{R}|_{R_0} = \beta[\tilde{R}, T]$. To n loop, it is a sum of terms of order $\tilde{T}^p \tilde{R}^{n+1-p}$, $0 \leq p \leq n$. The one-loop equation (i.e. adding $\tilde{T}\tilde{R}''$ to the first three terms in (13) below) exhibits the standard TBL for $u \sim \tilde{T}$ discussed in [8]. To 2-loop a term $\frac{-1}{4}\tilde{T}\tilde{R}''''(0)\tilde{R}''(u)$ appears, and using the TBL identity $\lim_{m\to 0} \tilde{T}\tilde{R}''''(0) = \tilde{R}'''(0^+)^2$, exact at one loop, produces precisely the 2-loop "anomalous" term in (13) below. Alas, one finds [15] that this procedure fails at 3-loop. One must instead examine the whole ERG hierarchy as in [8]. There, a method to obtain the unambiguous beta function was found by verifying order by order, a continuity property of the Γ-cumulants $S^{(n)}_{11..1}(u_{1..n})$ upon bringing points together. We completed in [15] the derivation of the (local) beta function, obtaining (up to a constant, with $\mathsf{R}'' = \tilde{R}'' - \tilde{R}''(0)$):

$$
\begin{aligned}
-m\partial_m \tilde{R} &= (\epsilon - 4\zeta)\tilde{R} + \zeta u \tilde{R}' + [\frac{1}{2}(\tilde{R}'')^2 - \tilde{R}''(0)\tilde{R}''] + \frac{1}{4}((\tilde{R}''')^2 - \tilde{R}'''(0^+)^2)\mathsf{R}'' \qquad (13) \\
&+ \frac{1}{16}(\mathsf{R}'')^2(\tilde{R}'''')^2 + \frac{3}{32}((\tilde{R}''')^2 - \tilde{R}'''(0^+)^2)^2 + \frac{1}{4}\mathsf{R}''((\tilde{R}''')^2\tilde{R}'''' - \tilde{R}'''(0^+)^2\tilde{R}''''(0^+)) \\
&+ \frac{1}{96}(\mathsf{R}'')^3(\tilde{R}^{(5)})^2 + \frac{3}{16}(\mathsf{R}'')^2\tilde{R}'''\tilde{R}''''\tilde{R}^{(5)} + \frac{1}{8}\mathsf{R}''((\tilde{R}''')^3\tilde{R}^{(5)} - \tilde{R}'''(0^+)^3\tilde{R}^{(5)}(0^+)) \\
&+ \frac{1}{16}(\mathsf{R}'')^2(\tilde{R}'''')^3 + \frac{9}{16}\mathsf{R}''((\tilde{R}''')^2(\tilde{R}'''')^2 - \frac{1}{6}R'''(0^+)^2(\tilde{R}'''')^2 - \frac{5}{6}\tilde{R}'''(0^+)^2\tilde{R}''''(0^+)^2) \\
&+ \frac{5}{16}((\tilde{R}''')^2 - \tilde{R}'''(0^+)^2)((\tilde{R}''')^2\tilde{R}'''' + \frac{1}{10}\tilde{R}''''\tilde{R}'''(0^+)^2 - \frac{11}{10}\tilde{R}''''(0^+)\tilde{R}'''(0^+)^2) + O(\tilde{R}^6)
\end{aligned}
$$

The first line are one and 2-loop terms, the second is 3-loop, the last three are 4-loop. Normal terms (i.e. non vanishing for analytic $R(u)$) are grouped with anomalous "counterparts" to show the absence of $O(u)$ term, a strong constraint (linear cusp, no supercusp): these combinations can hardly be guessed beyond 3 loop. This shows the difficulty in constructing the FT, already in $d = 0$. We emphasize that (13) results from *a first principle derivation.*

$R(u)$ being a physical observable, we look for cases where it can be computed. The Brownian landscape $V(u)$, the so-called Sinai model, is interesting as the $d = 0$ limit of random field disorder. Recently we obtained the full statistics of (deep) extrema in presence of a harmonic well [18]. This is generalized [15] as described in Fig 1. Graphically the renormalized landscape $\hat{V}(v) = E$ is constructed by raising a parabola P_v, $y(u) = -\frac{m^2}{2}(u - v)^2 + E'$ from $E' = -\infty$ until it touches (for $E' = E$) the curve $y = V(u)$ at point $u = u_1(v)$, position of the minimum of $H_{V,v}(u)$, E being the maximum (apex) of the parabola. P_v touching at two points $u_1(v) < u_2(v)$ signals a shock at $u = v$. Computing $P(E_1, v_1; E_2, v_2)$ (see Fig. 1) yields [15] (for $m^2 = 1$, $\sigma = 1$) and $\bar{R}(v) = R(v) - R(0)$:

$$\bar{R}(v) = -2^{\frac{1}{3}}\sqrt{\pi v}e^{-\frac{v^3}{48}}\int_{\lambda_1}\int_{\lambda_2}\frac{[1-\frac{2(\lambda_2-\lambda_1)^2}{b^2 v}]e^{i\frac{v}{2b}(\lambda_1+\lambda_2)-\frac{(\lambda_2-\lambda_1)^2}{b^2 v}}}{Ai(i\lambda_1)Ai(i\lambda_2)}[1+\frac{v\int_0^\infty dV e^{\frac{v}{2}V}Ai(aV+i\lambda_1)Ai(aV+i\lambda_2)}{Ai(i\lambda_1)Ai(i\lambda_2)}] \tag{14}$$

where $a = 2^{-1/3}$, $ba^2 = 1$, $\int_\lambda \equiv \int_{-\infty}^{+\infty}\frac{d\lambda}{2\pi}$ and all integrals converge well. One finds $\bar{R}(v) \approx -v + 0.810775$ at large v, and recovers [18] $\overline{u^2} = -R''(0) = 1.054238$. Once rescaled, (14) should be a FP of (13) corresponding to $\zeta = \zeta_{RF} = 4/3$.

At non-zero T, one reexamines (4) taking into account, within a droplet calculation, the probability density $D(y)dy$ for two degenerate minima of V, spatially separated by $y = u_2 - u_1$ ($D(y) = D(-y)$). It yields the TBL form (for $v \sim Tm^2$):

$$R''(v) = R''(0) + m^4 T\langle y^2 F_2(m^2 yv/T)\rangle_y \tag{15}$$

with $F_2(z) = \frac{z}{4}\coth\frac{z}{2} - \frac{1}{2}$ and $\langle ..\rangle_y \equiv \int dy..D(y)$ is normalized by the STS identity $\langle y^2\rangle_y = 2/m^2$. Since $F_2(z) \sim |z|/4$ at large z (15) yields consistent matching between finite T (droplet) quantities in the TBL and the cusp of the $T = 0$ FP for $v = O(1)$, with $R'''(0^+) = \frac{m^4}{2}\frac{\langle |y|^3\rangle_y}{\langle y^2\rangle_y}$. (15) should be more generally valid in $d = 0$ (any N), but in the RF case it is known [18] that (setting $m = \sigma = 1$) $D(y) = \frac{1}{2}\int_{\lambda_1,\lambda_2}\frac{Ai'(i\lambda_1)e^{i(\lambda_1-\lambda_2)|y|/b}}{Ai(i\lambda_1)Ai^2(i\lambda_2)}$ found to be consistent with $R'''(0^+) = 0.901289$ from (14). Remarkably, (15) generalizes to higher moments $\bar{S}^{(n)}$, with generalized functions F_n explicitly obtained in [15] yielding an exact "droplet" solution of the hierarchy (11) within the TBL.

Since in $d = 0$ the FRG (9) *identifies* with decaying Burgers, we emphasize the correspondence:

$$-R''(0) \equiv \overline{\mathsf{u}(x)^2} \quad , \quad \frac{T}{2}R''''(0) \equiv \nu\overline{(\nabla\mathsf{u}(x))^2} = \bar{\epsilon} \tag{16}$$

(more generally $\overline{\mathsf{u}(x)\mathsf{u}(0)} \equiv -R''(x)$), both have finite limits as $\nu \to 0$. The second is the *dissipative anomaly*: also present in 3D Navier Stokes. In Burgers it is due to shocks. The (equivalent) finite limit of the l.h.s. implies a thermal boundary layer in the FRG. Dilute shocks in Burgers are equivalent to droplets and a TBL in the FRG where $u_{21} =\equiv \mathsf{u}(0^+) - \mathsf{u}(0^-)$, and (15) can indeed be recovered from a single shock solution in Burgers upon averaging over its position [15]. The celebrated Kolmogorov law in the inertial range:

$$\frac{1}{2}\bar{S}_{111}(0,0,u) \sim \bar{\epsilon}u \equiv \frac{1}{12}\overline{(\mathsf{u}(x)-\mathsf{u}(0))^3} \sim -\bar{\epsilon}x \tag{17}$$

corresponds to the non-analytic behaviour of the third cumulant at small argument in the $T = 0$ theory. Identical coefficients in (16) and (17) are a consequence of matching across the TBL (i.e. viscous layer), identifying the second derivative of (10) at $v = 0$ (for $\nu > 0$) and $v = 0^+$ (for $\nu \to 0$) i.e $\partial_t R''(0) = TR''''(0) \equiv \partial_t R''(0^+) = \bar{S}_{112}(0,0,0^+)$. Similar relations exist in stirred Burgers (and Navier Stokes) [10]: there the dissipation rate $\bar{\epsilon}$ is balanced by forcing, instead of scale-invariant time decay of correlations, but small-scale shock properties should be rather similar. Closure of hierarchies similar to (11) was proposed there [13] in terms of an "operator product expansion". Recent studies cast doubt on such simple closures [11]: N=1 decaying Burgers (and stirred [14]) can be constructed in the inviscid limit ($\nu \to 0$) using distributions, e.g. $tF'(v) = 1 - \sum_s u_{21}^{(s)}\delta(v - v_s)$. It is shown there that shock "form factors" (i.e. size-distribution) determines small distance (non-analytic) behaviour of moments of velocity differences, $\overline{(F(v)-F(0))^p} \sim \mu_p v\, sign(v)^{p+1}$, with $\mu_p = \overline{\sum_s (u_{21}^{(s)})^p\delta(v - v_s)}$. In the

FRG these are equivalent to droplet distributions as we show [15] that $\mu_p = \langle |y|^{p+1} \rangle_y / \langle y^2 \rangle_y$, e.g. consistent with $R'''(0^+) \equiv \overline{\mathsf{u}(0^+)\nabla \mathsf{u}(0)} = \mu_2/(2t^2)$ given above. The $T=0$ distributional limit of (9) derived in [11] is equivalent to $\partial_t \hat{V}(v) = -\frac{1}{2}F(v^+)F(v)$: it validates the first-principle FRG discussed above yielding (13) (the central property being continuity of all $\bar{S}_{1..1}$ since $F(v)$ remains bounded). These considerations should be universal for dilute shocks, i.e. independent of details of shock probability distributions. For RF disorder the full distribution of shock parameters $\{u_{21}^{(s)}, v_s\}$ is known exactly [19]. It is used in [15] to obtain from (7) another expression for $\Delta(v)$ fully consistent with (14).

Extensions to higher d are studied in [15]. Let us indicate here that similar droplet estimates can be performed and yield an exact solution of the full *functional* hierarchy (12) for all moments. Within the TBL $m^2 v_x/T = O(1)$ the $R[v]$ functional reads:

$$R[v] = \frac{1}{2} v_x R''_{xy}[0] v_y + T^3 \sum_i \langle H_2 (\int_{xy} v_x g_{xy}^{-1} u_{12,y}^{(i)} / T) \rangle_D \tag{18}$$

where $H_2''(z) = F_2(z)$. To find this solution one considers a small density (of order Tm^θ) of well separated "elementary droplets", i.e. local GS degeneracies $u_{12,x}^{(i)} = u_{2,x}^{(i)} - u_{1x}$. $\langle .. \rangle_D$ denotes the average over them. Eq. (18) relates droplet probabilities to the TBL in the FRG.

To conclude we related FRG functions, e.g. $R(u)$, to observables. This allows to compute them in simple cases, and provides a method to measure them in numerics and experiments. Their relations to shocks in energy (or force) landscape was made precise, via a generalized Burgers equation. We have shown how shock form factors and droplet distributions are related to the FRG functions. Questions such as the extent of universality in the TBL, how do properties of $N=1$, $d=0$ Burgers extend to functional shocks (e.g. Kolmogorov law) remain tantalizing but can now be adressed.

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